Diamonds and Dominoes.

Impossibility Results for Associative Modal Logics

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We’ll begin with something well known (and then something that, I think, deserves to be better known)

Appetizer

Something well known: Classical Propositional Logic is decidable.

Let's add another connective $\circ$.

What axioms and rules should govern $\circ$? Let's say:

- $\varphi \circ \bot \to \bot$, $\bot \circ \varphi \to \bot$,
- $\varphi \circ (\psi \vee \chi) \leftrightarrow (\varphi \circ \psi) \vee (\varphi \circ \chi)$, $(\psi \vee \chi) \circ \varphi \leftrightarrow (\psi \circ \varphi) \vee (\chi \circ \varphi)$,
- $(\varphi \circ \psi) \circ \chi \leftrightarrow \varphi \circ (\psi \circ \chi)$,
- $$\frac{\varphi \leftrightarrow \varphi'}{\varphi \circ \psi \leftrightarrow \varphi' \circ \psi}, \qquad \frac{\varphi \leftrightarrow \varphi'}{\psi \circ \varphi \leftrightarrow \psi \circ \varphi'}.$$

Question: Is the resulting system decidable?

Answer: It is not! (cf. Kurucz, Némethi, Sain, and Simon 1995)

And in fact, it is the modal logic

$$\mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$$

Plan for the rest of the talk

- Setting
- Results and technique
- Related results

Preprint available on arXiv.

Setting

Logically, we are interested in normal extensions of

$$\mathbf{AK}_2 := \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r).$$

Algebraic semantics for $\mathbf{AK}_2$ is given by *associative BAOs* $(A, \vee, \wedge, \neg, \bot, \top, \circ)$:

- $(A, \vee, \wedge, \neg, \bot, \top)$ is a BA
- $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
- $x \circ \bot = \bot = \bot \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$.

Relational semantics for $\mathbf{AK}_2$ is given by *associative frames* $\mathbb{F} = (X, \cdot)$:
$\cdot : X^2 \to \mathcal{P}(X)$ is a function s.t.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and

$$\mathbb{M}, x \Vdash \varphi \circ \psi \quad \text{iff} \quad \text{there exist } y, z \in X \text{ such that } \mathbb{M}, y \Vdash \varphi; \ \mathbb{M}, z \Vdash \psi; \text{ and } x \in y \cdot z.$$

$\cdot$ is lifted to sets $Y, Z \subseteq X$ by $Y \cdot Z := \{x \in X \mid \exists y \in Y, z \in Z : x \in y \cdot z\}$.

Two central systems

1. Take frames $(X, \cdot)$ to be semilattices: $\cdot$ is functional, associative, commutative, and idempotent.
   - We obtain the modal logic of semilattices (decidability problem raised by SBK (2023a)).
   - Algebraically, this is $\mathsf{Var}(\mathsf{SL}^{+})$ (raised by Bergman (2018) and Jipsen, Eyad Kurd-Misto, and Wimberley (2021)).
2. Take frames $(X, \cdot)$ to be Boolean algebras (raised by Goranko and Vakarelov (1999)).[1]

Goal: Prove (un)decidability.

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

The Domino Problem

- A Wang domino (tile) is a square with colors on each side.
- The domino (tiling) problem: Given a finite set of Wang tiles $\mathcal{W}$, is it possible to tile the first quadrant $\mathbb{N} \times \mathbb{N}$ so that adjacent tiles match along their shared edges
- Introduced by Wang (1963) and proven undecidable by Berger (1966).

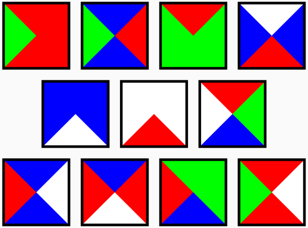


Figure 1: Wang tiles

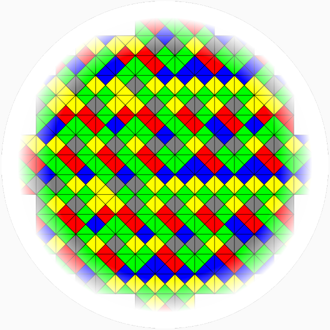
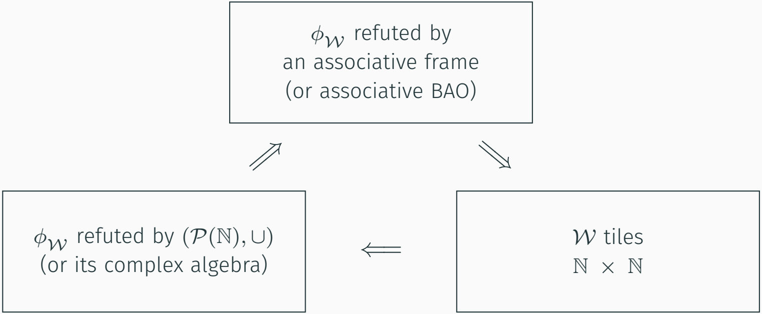


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

Main theorem

Given $\mathcal{W}$, construct a formula $\phi_{\mathcal{W}}$ such that:

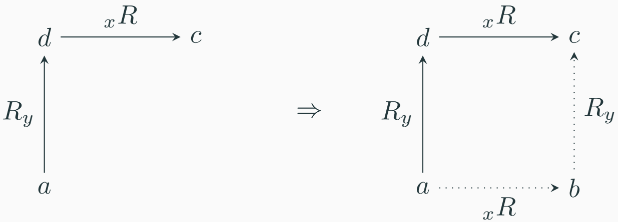


Theorem

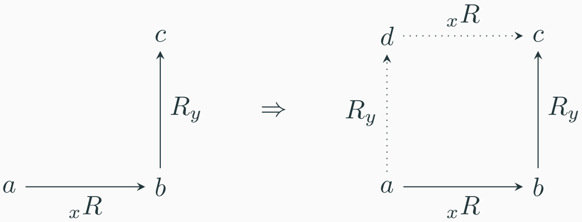
Let V be a variety of associative BAOs.
If V contains $(\mathcal{P}(\mathbb{N}), \cup)^+$, then V is undecidable.

Let $\mathbf{L}$ be an associative normal modal logic.
If $\mathbf{L} \subseteq \text{Log}(\mathcal{P}(\mathbb{N}), \cup)$, then $\mathbf{L}$ is undecidable.

Associativity and Tiling 1



From ($Rady$ and $Rdxc$) infer $\exists b \in X$ ($Raxb$ and $Rbcy$).



From ($Raxb$ and $Rbcy$) infer $\exists d \in X$ ($Rady$ and $Rdxc$). [2]

[2] ${}_zR := \{(a,b) \in X^2 \mid Razb\}$ and $R_z := \{(a,b) \in X^2 \mid Rabz\}$

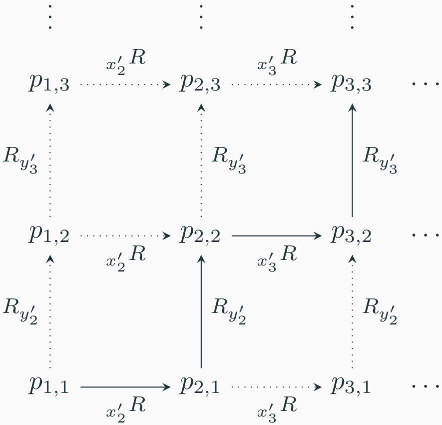


Figure 3: Generating $\mathbb{N}^2$ from the staircase: $p_{1,1}, p_{2,1}, p_{2,2}, p_{3,2}, p_{3,3}, \ldots$.

Consequences

Theorem

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Recall the above. From this, we get:

Theorem

$\text{Var}(\text{BA}^+)$ is **undecidable**.

Hyperboolean modal logic is **undecidable**.

Theorem

$\text{Var}(\text{SL}^+)$ is **undecidable**.

The modal (information) logic of semilattices is **undecidable**.

Proof. Semilattices are associative and $(\mathcal{P}(\mathbb{N}), \cup)$ is, in particular, a semilattice.

Other Consequences

Undecidability for:

- The variety of Boolean semilattices (Bergman 2018 and Jipsen, Eyad Kurd-Misto, and Wimberley 2021).
- Modal logics over (modular/distributive) lattices (Wang and Wang (2025)).
- Conservative extension of Skvortsov's modal logic.

New undecidability proofs for:

- $\mathbf{AK}_2 = \mathbf{K}_2 \oplus (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$ [Kurucz, Némethi, Sain, and Simon 1993, 1995]
- The classes of algebras isomorphic to (commutative) algebras of binary relations closed under composition, intersection ($\cap$), union ($\cup$), and complementation (^{c}) [Hirsch, Hodkinson, and Jackson 2021, Cor. 11.3]
- Boolean Bunched Implication logic (BBI) [Brotherston and Kanovich 2010; Kurucz, Némethi, Sain, and Simon 1995; Larchey-Wendling and Galmiche 2010]

Lastly: There is no translation from modal information logic into truthmaker semantics (question raised by van Benthem 2017, 2024)

Related results

Thank you!

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Team semantics as relational semantics

For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	**iff**	$\forall v \in s : v(p) = 1$,
$s \vDash \varphi \wedge \psi$	**iff**	$s \vDash \varphi$ *and* $s \vDash \psi$,
$s \vDash \varphi \mathbin{\rotatebox[origin=c]{180}{$\bigwedge$}} \psi$	**iff**	$s \vDash \varphi$ *or* $s \vDash \psi$,
$s \vDash \sim\varphi$	**iff**	$s \nvDash \varphi$,
$s \vDash \varphi \vee \psi$	**iff**	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

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$s \vDash {\sim}\varphi$	**iff**	$s \nvDash \varphi,$
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$s \vDash \varphi \circ \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$;

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For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{or} \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.